Gordon (*The Century*, January, 1898), as follows: "The numbers from one to four are represented by dots; a bar signifies five; a bar and a dot six; ten is written by two bars; and so on up to nineteen, the sign of which is three bars and four dots; after this number the signs employed are in doubt." The Egyptians used tally marks up to nine but ten is an inverted U. Cuneiform numerals are also tally marks. Even as late as the third century B. C. in India the Asoka edicts record numbers up to five as vertical tally marks.

In the accompanying chart, there is some evidence that, as a rule, tally marks originally went only to five after which the one or five was repeated with extra markings. Professor Edwin S. Crawley (*Popular Science Monthly*, August, 1897), says that all systems of numbers were originally quinary from the use of the fingers of one hand in counting and became decimal as soon as two hands were used, but others state that there is no

INDIAN.

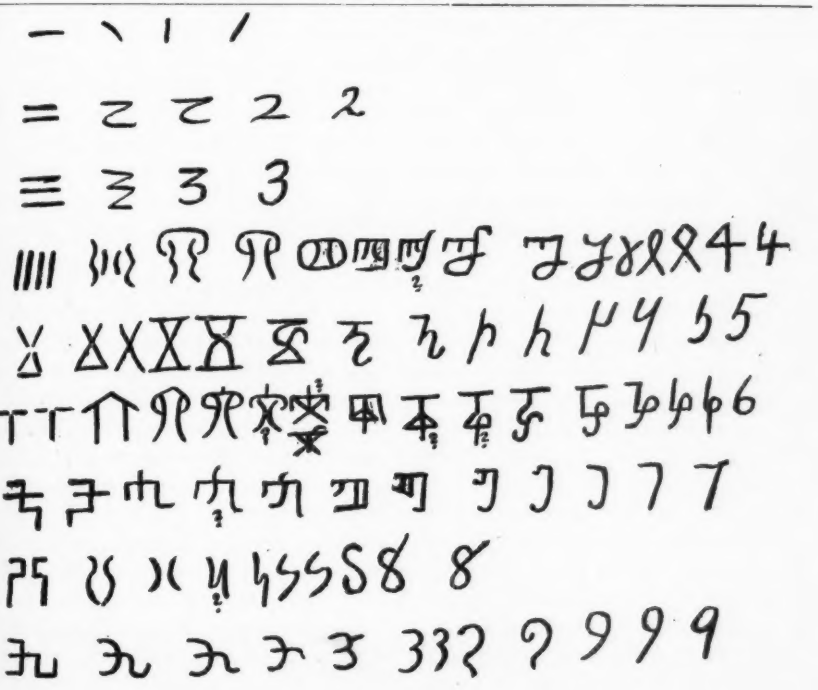
Jaina Mss.	Jaina Mss.	Nepal Mss.	"Bower" Mss.	Old Nepal Handwriting.	"Old Forms."	Eastern Caves.	Western Caves.	Bactrian.	"Very Old."	First Century.	Fifth Century.
১	১	১	১	১		—	—	—	—	—	⌒
	২	২	২	২		=	=	=	=	=	⌒⌒
	३	३	३	३		≡	≡	≡	≡	≡	⌒⌒⌒
৩	৩	৩	৩	৩	⊀	⊀	⊀	⊀	⊀	⊀	⊀
৪	৪	৪	৪	৪	⊢	⊢	⊢	⊢	⊢	⊢	⊢
৫	৫	৫	৫	৫	⊢	⊢	⊢	⊢	⊢	⊢	⊢
৬	৬	৬	৬	৬	⊢	⊢	⊢	⊢	⊢	⊢	⊢
৭	৭	৭	৭	৭	⊢	⊢	⊢	⊢	⊢	⊢	
৮	৮	৮	৮	৮	⊢	⊢	⊢	⊢	⊢	⊢	
৯	৯	৯	৯	৯	⊢		⊢	⊢	⊢	⊢	
০	০	০	০			∝	∝	∝	∝		⌒

foundation for such a generalizetion. A few people have used both hands and feet, a vigesimal system, whose remnants are seen in the "quatre-vingts" of the French from 81 to 99 and our "score" of one finished tally group. Duodecimal systems were sometimes grafted on as in Babylonia, making also a sexagesimal system, side by side with the decimal. The Chinese numerals for 11 to 20 are merely those from one to ten, with a ten (X) over or before them, but the commercial forms are quinary, the six to eight being based on the one to five with extra tally marks beneath.

The numerals in the accompanying chart are copied from Isaac Taylor's "The Origin of the Alphabet," from the pamphlet on the subject by Professor John C. C. Clark, and from a paper by Karl Mischke (*Deutsche Japan Post*, Yokahama, Japan, December 29, 1906). Edward Clodd, "The

Story of the Alphabet," derived his from Taylor, who in 1882, described the changes in Arabic numerals. The Chinese forms are copied mostly from the monograph on early Chinese writing by Rev. Frank H. Chalfant, a missionary in China (*Memoirs of the Carnegie Museum*, Pittsburg, Vol. IV, No. 1), to which have been added a few old forms obtaned through the courtesy of Mr. L. C. Solyom of the Congressional Library, who copied them from "Luh-shoo-fun-luy," a dictionary of ancient forms and characters, by Foo Lwan-tseang. Others were obtained through the kindness of Professor Friedrich Hirth, Professor of Chinese in Columbia University in New York, who searched through Chinese archeological inscriptions for the purpose. Two columns of Nepal and Jaina manuscript and the "Bower" manuscript are copied from George Bühler's "Indische Paleographie," in which there are also a large number of variants, so closely resembling those copied elsewhere that they have been omitted here. Identical forms are also found in Burnell's "South Indian Paleography." There are also an enormous number of Medieval European forms in Cappelli's "Dizionario di Abbreviature Latine ed Italiane" and in Chassant's "Dictionaire des Abréviations Latines et Française" but they are almost identical with the ones here given. This infinite variety is partly due to an occasional scribe's fondness for decorating

PROBABLE EVOLUTION OF THE NUMERALS.



with flourishes, as well as due to the gradual evolution of simple forms from more complex ones. In all of them there must also be considerable allowance made for incorrect transcribing in the numerous times they have been copied from one book to another since the original was taken from some manuscript, coin, monument, or vase, and some were probably erroneously translated.

Tally marks must be so distinctive that they can be read, no matter how the coin or stick is held, vertically, horizontally, or inverted. The modern six and nine would have been useless, for they would have been indistinguishable unless the top was known. Consequently we find many instances in which the numerals have been inverted or inclined at various angles as it was of no practical importance at first whether they were upside down or not. Our two, for instance, as well as the four, five, six, or seven have all been inverted and reversed at times. There may also have been reversals of the forms due to the fact that the people from whom the Arabs obtained the numerals, probably wrote from left to right while the Arabs wrote from right to left, and the Europeans from left to right. In addition to this, one Sanskrit language was written from right to left, and another from left to right. It is remarkable that the old Arabic forms have changed less in Europe than in Arabia. The final European forms of 6, 7, 8, 9, and 10 after many vicissitudes of change are nearly identical with old Arabic forms and show strong relationship to the early Indian. The vast majority of the old forms are like fossils of extinct species which have left no descendants; but by picking out forms here and there, it is possible to secure a gradation of changes from the first to the last, the zero, by the way, being a very late development, possibly from the position dot of the Arabic, which in turn was probably a modification of the Indian O or ◌. The use of position of some sort to indicate multiple value is a recent invention, G. R. Kaye stating that there is no evidence of position value prior to 1000 A. D. (*Journal Royal Asiatic Society*, July, 1907).

The numeral one, is always a single tally mark, straight or curved, horizontal, vertical, or at an angle. Two was evidently two tally marks and cursively became the modern 2 or the reversed or inverted form. Three in all its forms is evidently derived from three parallels. But the higher numerals have not so clear an origin. It is quite difficult to distinguish groups of parallel lines of more than three. We use a cross tally for every fifth stroke making groups of five, and probably we once grouped a "score" together also to indicate each time all the fingers and toes were counted. The Egyptians used groups of three strokes up to nine. Consequently some other system than parallel groups must have been used for the original numerals of four to ten. The modern Arabic has even returned to figures of four and five strokes for these two numerals, after having evolved other forms which they later abandoned, and there seems often to be an effort in ancient Indian forms of four and five to make them of four and five strokes, respectively.

The evolution of the Chinese numerals from tally marks is evident from a mere glance at the chart, for in every case there is a gradation to the modern. There are several facts which must be kept in mind in interpreting these changes. In the first place there are two evolutions recorded side by side — the oldest or quinary system which developed into the modern commercial, and the later decimal which formed the common numerals. Both of these are exceedingly ancient in origin, possibly a thousand years before the Christian era or even much earlier still for they had developed into the present decimal forms by 200 A. D. To represent six, seven, eight, and nine, the Chinese evidently first used five vertical tally marks under which they placed one to four horizontals, respectively, but the five verticals were early contracted into one. The commercial forms for six, seven, and eight are thus exceedingly old. The nine was evidently abandoned for the more convenient decimal system. The X which represents the commercial four is a direct descendant of the four vertical tally marks, the horizontal marks were used for the numeral four as late as the Ham dynasty, 206 — 220 A. D. After this time only the new form appears though its origin was probably very much more remote. The commercial five which looks like our eight, is also a direct descendant of a very old form of five strokes which early replaced the five parallel ones — the transition forms given by Chalfant being quite conclusive as to this point.

When a decimal system became necessary, the Chinese, or those from whom they borrowed, had to invent other forms of tally marks for the numerals from five to nine, as it was clearly impracticable to continue the use of parallel strokes, but why they should have hit upon the curious forms recorded is a mystery.

Another fact must be kept in mind in explaining some of these changes. The use of a brush for a pen made it impossible to write the old curved forms which were carved on stone or written with a solid pencil. Chinese writing thus very early abandoned curves for straight lines, — the squares and angles of the modern writing. This fact explains the change from the rounded forms of 100 A. D. into the modern numerals at about that date. The modern four, for instance, is sometimes thought to have been four vertical strokes inclosed in a square, and that it changed into the modern form of a square containing two short strokes. The process was the opposite, for the square is the manner in which the oval of the older form had to be written with a brush, and the oval form is evidently a descendant of the earlier vertical four strokes. The Chinese "running" characters for two and three are the same as our own. The standard numerals (Chêng-tzu) are not given on the chart as they are complicated late inventions to prevent swindling, the commercial and common numerals being very easily changed to higher ones by the addition of a stroke or two. The human figure or a weapon of some sort seems to have been used in addition to the tally marks in the original decimal numerals. Bühler (*Journal Royal Asiatic*

Society, 1882) says that there is no doubt that the numerals were introduced into India from without, as they appear rather suddenly and in a well evolved form. The original forms, then, are not found in India and the oldest known Indian must be evolutionary forms of some nearby Mongolian system, though Bühler and many others are inclined to believe that they came from Egypt. Prior to their introduction the Hindoos used quite a variety of methods of expressing numbers which do not concern us here. Though letters and syllables were used, there are others having no such derivation. Indeed one writer, Kern, is quoted by Bühler as believing the old Indian four and five to be four and five strokes, respectively. The identity of the first three numerals in Chinese and Indian would lead us to suspect identity of origin of the two systems, or more probably that the Chinese forms were carried by traders into India. Indeed Dr. Fritz Hommel ("The Civilization of the East") states that Indian "Culture is in the main an offshoot of the Chinese." The Chinese would also be suspected on account of the identity of the old Nepal nine and an evolutionary Chinese nine and there is a perfect gradation of forms from the oldest Chinese nine tally marks to our present nine.

Doubtless there was a considerable interval between the date of introduction of the numerals into India and the date of the oldest surviving forms —an interval in which their origin was forgotten if it was ever known. This fully accounts for the fact that in some cases considerable change had taken place. Probably many of the ancient Indian evolutionary forms will never be found, nevertheless those now known are within the bounds of ordinary variation of writing. They were undoubtedly introduced while some still retained an evident form of tally marks, which led to the frequent writing of four and five with four and five strokes, respectively. Some of the old Indian fives are almost identical with the Chinese original form, and in the six there are undoubted affinities, the sevens are closer still and nines identical. Moreover these resemblances are between forms of practically the same period. The well evolved "western cave" forms are ascribed to the first and second century A. D., and the older Indian must be set at least two centuries earlier which makes them contemporaneous with the later Chinese evolutionary forms which they resemble. Indian dates prior to the Christian era are very uncertain, for it is said the Brahma laws were not written until about 325 B. C. As the Asoka edicts did not have the numerals in 250 B. C., and as the cave numerals were well evolved forms in the first century A. D., we have a fairly definite period in which they were introduced and evolved.

Among the old Indian forms given by Kaye (*Journal Asiatic Society of Bengal*, July, 1907) there is an eight which is evidently an attempt to write the Chinese eight cursively, and it makes this evolutionary series almost as complete as the nine. The interesting thing about our eight and two, is that each seems to be the result of an attempt to write two parallel

strokes cursively, the two was two horizontal parallel bars and the eight two curved vertical lines, and some of the old forms of eight are practically the same as the two turned over 90°.

It is quite likely also that when forms of numerals are evidently tally marks, the ancient tribes would not stick to any particular arrangement, but form new ones provided they indicated numbers. This is the most reasonable explanation of the very evident tally-mark nature of the numerals in the Jaina manuscript. Ten is a nine with an extra stroke, and the eights are sevens with an extra stroke. The Jaina four, five, and six are also clearly derived from groups of marks. In course of time, by slurring, omission of strokes and adding embellishing flourishes, the Nepal and Bower manuscript forms arose. Indeed in the seven there is a perfect gradation of evolutionary forms to our present seven. In the four the resemblance is seen by making an assumption. In the five there is more evidence of an attempt to write cursively one of the X forms of the Chinese, but the six is not so evident without making two assumptions.

The supremacy of the Chinese numerals is explained by the fact that they were the first ideographs in the field. Egyptian pictographs evolved in the direction of representing sounds and, besides, their tally marks elsewhere were in groups of parallels, and not the fortunate Chinese groupings which lent themselves to change into ideographs. The invention of position value of course killed all the numerals above nine.

ON THE NUMBER OF EQUAL REGULAR SPHERICAL POLYGONS THAT CAN BE CONSTRUCTED SO AS TO COMPLETELY COVER A SPHERE.

By B. F. YANNEY, Alliance, Ohio.

Let N=the number of equal polygons required; n=the number of sides in each, and k=the number of angles about each common vertex.

Then will $[\frac{360}{k}n-(n-2)180]N$=the area, in spherical degrees, of the sum of all the spherical polygons completely covering the sphere; but this area is also equal to 720.

Therefore, $[\frac{360}{k}n-(n-2)180]N=720$; whence, $N=\frac{4k}{2n-nk+2k}$.*

It remains to solve this equation for positive integers.

*This formula may be found on page 69, Vol. III, of Henrici and Treutline's *Geometry*, though developed by a different method, and having diffe.ent considerations in view.

1. For $k=1$, $N=\frac{4}{n+2}$. Now, the only positive integral value n can have, to make N likewise integral, is 2; from which $N=1$. This is the case of a lune with its sides coincident and its angle 360°.

2. For $k=2$, $N=2$ for any value of n. This is the case of two hemispheres, each of which may be considered as bounded by any number of sides, each angle being an angle of 180°. In particular, n may equal 2, in which case each hemisphere is regarded as a lune.

3. For $k=3$, $N=\frac{12}{6-n}=3$, 4, 6, or 12, according as $n=2$, 3, 4, or 5, respectively. The figures are, in order, lunes, equilateral triangles, regular quadrangles, and regular pentagons.

4. For $k=4$, $N=\frac{8}{4-n}=4$ or 8, according as $n=2$ or 3. The corresponding figures are lunes or equilateral triangles.

5. For $k=5$, $N=\frac{20}{10-3n}=5$ or 20, according as $n=2$ or 3. The corresponding figures are lunes or equilateral triangles.

6. For $k=6$, $N=\frac{24}{12-4n}=6$, when $n=2$. [Lunes.] When $n=1$, $N=3$. But $k=6$, $n=1$, $N=3$ are incompatible with the nature of the problem.

7. For $k>6$, $n=2$, and $N=k$. From $N=\frac{4k}{2n-nk+2k}$, it is easily seen that, for $n=2$, $N=k$. This includes, as well, the six cases of lunes already considered. From the denominator $2n-nk+2k$ it is seen that $2n+2k$ must be greater than nk: $2n+2k>nk$.

Therefore, $n(k-2)<2k$, whence $n<\frac{2k}{k-2}=2+\frac{4}{k-2}$. Now it is clear that for $k>6$, $n<3$. That is, n must be 2 or 1. But the values of n have both been considered.

Table of Results:

$k=1$, $n=2$, $N=1$ . . .	Lune.
$k=2$, n, any value, $N=2$	Hemispheres.
$k=3$, $n=2$, $N=3$ . . .	Lunes.
$k=3$, $n=3$, $N=4$ . . .	Triangles...A.
$k=3$, $n=4$, $N=6$ . . .	Quadrangles...B.
$k=3$, $n=5$, $N=12$ . . .	Pentagons...C.
$k=4$, $n=2$, $N=4$ . . .	Lunes.
$k=4$, $n=3$, $N=8$ . . .	Triangles...D.
$k=5$, $n=2$, $N=5$ . . .	Lunes.
$k=5$, $n=3$, $N=20$ . . .	Triangles...E.
$k>6$, $n=2$, $N=k$ . . .	Lunes.

From A, B, C, D, and E, we may easily pass to the proof that there are five and only five regular curvex polyhedrons.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

316. Proposed by B. F. FINKEL, Ph. D.

Prove that $\sum_{r=1}^{r=n}(-1)^{n-1}\frac{1}{n}{}_nC_r=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$ where ${}_nC_r=\frac{n(n-1)\ldots(n-r+1)}{1.2.3\ldots r}$. Dickson's *College Algebra*, ex. 13, p. 92.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

The first member of the equation should be $\sum_{r=1}^{r=n}(-1)^{r-1}\frac{1}{r}{}_nC_r$ instead of $\sum_{r=1}^{r=n}(-1)^{n-1}\frac{1}{n}{}_nC_r$.

$$S_n=\sum_{r=1}^{r=n}(-1)^{r-1}\frac{1}{r}{}_nC_r=n-\frac{n(n-1)}{2.2!}+\frac{n(n-1)(n-2)}{3.3!}-\text{ to } n \text{ terms.}$$

$$S_{n+1}=(n+1)-\frac{(n+1)n}{2.2!}+\frac{(n+1)(n-1)}{3.3!}-\text{ to } n+1 \text{ terms.}$$

$$\therefore S_{n+1}-S_n=1-\frac{n}{2!}+\frac{n(n-1)}{3!}-\text{ to } n+1 \text{ terms}=\frac{1}{n+1}[1-(1-1)^{n+1}]=\frac{1}{n+1}.$$

$$S_1=1,\ S_2-S_1=\tfrac{1}{2} \text{ or } S_2=1+\tfrac{1}{2},$$
$$S_3-S_2=\tfrac{1}{3} \text{ or } S_3=1+\tfrac{1}{2}+\tfrac{1}{3},$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$S_n-S_{n-1}=1/n \text{ or } S_n=1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots+1/n.$$

(Vol. VII, No. 4, of the MONTHLY, page 105, line 8, gives $S_{n+1}-S_n=\frac{1}{1+n}$ by substituting -1 for x.)

Also solved by S. Lefschetz.

317. Proposed by FRANCIS RUST, Allegheny, Pa.

Once, in classic days, Silenus lay asleep; a goat skin filled with wine near him. Dionysius passing by, profited, by siezing the skin, and drinking for two-thirds ($\frac{2}{3}$) of that time in which Silenus alone could have emptied said skin. At this point Silenus awoke, and seeing what was happening, snatched away the precious skin, and finished it.

Now, had both started together, and drank simultaneously, they would have consumed the wine skin in two hours less time. And, in this case, Dionysius' share would have been one-half as much as Silenus did secure, by waking and snatching the skin.

In what time would either one of them alone finish the goat-skin?

Solution by PROFESSOR F. L. GRIFFIN, Ph. D., Williams College.

Let x=fractional part which S drank, and y=number of hours S requires for entire skin. Then $\frac{2}{3}y$=time D was drinking, and xy=time S was drinking; $y(\frac{2}{3}+x)$ = time they used consecutively. Also, since $\frac{3(1-x)}{2y}$=part D drinks per hour, or $\frac{3(1-x)+2}{2y}$=part both drink per hour, the time required when drinking simultaneously $=\frac{2y}{5-3x}$.

Hence, (A) $y(\frac{2}{3}+x)=2+\frac{2y}{5-3x}$.

Again, the part D would get when they drink simultaneously $=\left(\frac{2y}{5-3x}\right)\frac{3(1-x)}{2y}$, or $\frac{3-3x}{5-3x}$; hence, by the problem, (B) $\frac{3-3x}{5-3x}=\frac{1}{2}x$.

Equation (B) gives $x=\frac{2}{3}$ or 3, the latter value being impossible.

Then (A) becomes $\frac{4}{3}y=2+\frac{2}{3}y$, or $y=3$; and since D drinks $\frac{1}{6}$ per hour his time would be 6 hours.

Also solved by V. M. Spunar, G. B. M. Zerr, and J. Scheffer.

318. **Proposed by PROFESSOR R. D. CARMICHAEL, Anniston, Ala.**

Sum to infinity the series $n/(4n^2-1)^2$ beginning with $n=1$.

Solution by J. W. CLAWSON, Ursinus College, Collegeville, Pa.; HOWARD C. FEEMSTER, A. B., York College, York, Neb.; J. EDWARD SANDERS, Weather Bureau, Chicago, Ill., and S. LEFSEHETZ, Pittsburg, Pa.

$$\sum_{n=1}^{n=\infty}\frac{n}{(4n^2-1)^2}=\sum_{n=1}^{n=\infty}\frac{1}{8}\left[\frac{1}{(2n-1)^2}-\frac{1}{(2n+1)^2}\right]$$

$$=\underset{n=\infty}{\text{Lt.}}\frac{1}{8}\left[\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\frac{1}{(2n-1)^2}\right)\right.$$

$$\left.-\left(\frac{1}{3^2}+\frac{1}{5^2}+\ldots\frac{1}{(2n-1)^2}+\frac{1}{(2n-1)^2}\right)\right]=\underset{n=\infty}{\text{Lt.}}\frac{1}{8}\left[\frac{1}{1^2}-\frac{1}{(2n+1)^2}\right]=\frac{1}{8}.$$

Also solved by V. M. Spunar, G. B. M. Zerr, J. Scheffer, S. A. Corey, and T. J. Fitzpatrick.

GEOMETRY.

343. **Proposed by O. J. BROWN, Fairhope, Ala.**

From any external point of a triangle, to draw a line so as to divide the triangle into two equal parts.

I. Solution by C. N. SCHMALL, New York City.

Construction: Let P be the given point and ABC the given triangle. Join D, E, F, the middle points of the sides. Now draw PGI parallel to BC and join EG. From D draw DH parallel to GE, and then draw HI parallel to BA and meeting PG prolonged in I. On PI as a diameter describe a semi-circle and thereon lay off $PK=PG$. Draw KI, and in the base BC lay of $HM=KI$. Draw PM cutting AB in L. The line PLM bisects the triangle and is the line required.

Proof: Let PM and HI meet in N. Now the triangles PGL, PIN, MHN are clearly similar.

Also, since $PI^2=PK^2+KI^2=PG^2+HM^2$ (for $PK=PG$, and $KI=HM$).

$\therefore \triangle PIN=\triangle PGL+\triangle MHN$...(1).

Hence, the quadrilateral $LGIN$ is equal to the triangle MHN;

$\therefore$ triangle BML=parallelogram $BHIG$...(2).

Again, GE and DH are parallel;

$\therefore GB : BE = DB : BH$...(3).

Hence, the parallelograms $BHIG$ and $BEFD$ have a common angle B, and the including sides are reciprocally proportional.

$\therefore$ by (3), parallelogram $BHIG$=quadrilateral $BEFD$.

But by (2), parallelogram $BHIG$=triangle BML...(2).

$\therefore$ triangle BML=parallelogram $BEFD=\frac{1}{2}$ triangle ABC.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let P be the given point. Draw PE parallel to AB, cutting AC in D; make parallelogram $DEFA=\frac{1}{2}$ triangle ABC. On AB at F erect perpendicular $FG=PD$, and make $GA=PE$. Connect P with Q, then PQ will be the required line which bisects triangle ABC.

For, $\triangle FHQ : \triangle PHE=FQ^2 : PE^2=PE^2-PD^2 : PE^2$.

$$\therefore \frac{\triangle FHQ}{\triangle PHE}=1-\frac{PD^2}{PE^2}=1-\frac{\triangle PDH}{\triangle PEH}$$

$\therefore \triangle FHQ=\triangle PEH-\triangle PDI=DIHE$; $\therefore FHQ+IHFQ=DIHE+IHFA=DEFA$.

Note: To construct $DEFA$, connect C with the mid-point M of AB, draw DM, and CN parallel to DM, then F will be the mid-point of AN.

Also solved by G. B. M. Zerr, V. M. Spunar, and Daniel B. Northrup.

NOTE. The following gentlemen should have received credit for solving 342: J. A. Caparo, V. M. Spunar, and S. Lefschetz.

CALCULUS.

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274. Proposed by J. EDWARD SANDERS, Weather Bureau, Chicago, Ill.

About the vertices of a regular tetrahedron four spheres are drawn with radii equal to the edge of the tetrahedron. Find the volume common to them all.

Solution without the use of the Calculus by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

One vertex of the tetrahedron is at the center of each sphere while the other three vertices are in the surface.

Hence the total volume common to all the spheres = the volume of the tetrahedron + the volumes of four equal spherical segments.

Let a = the edge of the tetrahedron and also the radius of each sphere. Then $\frac{1}{3}a\sqrt{6}$ = altitude of tetrahedron, and $\frac{1}{12}a^3\sqrt{2}$ = its volume.

$a-\frac{1}{3}a\sqrt{6}=\frac{a}{3}(3-\sqrt{6})$ = altitude of segment. $\frac{\pi a^3}{81}(3-\sqrt{6})^2(6+\sqrt{6})=$

$\frac{\pi a^3}{27}(18-7\sqrt{6})$ = volume of spherical segment.

Total volume required $= V=\frac{4\pi a^3}{27}(18-7\sqrt{6})+\frac{a^3\sqrt{2}}{12}$.

$\therefore V\frac{a^3}{108}(9\sqrt{2}+288\pi-112\pi\sqrt{6})$.

This solid might be called a spherical tetrahedron.

275. Proposed by C. N. SCHMALL, 604 East 5th Street, New York City.

Explain fully why the circular measure of an angle is used in the calculus.

No satisfactory answer of this problem has been received.

276. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

In a certain country the tax per \$1 on a person's income varies as the cube root of the number of dollars, and when the income is \$8000 the rate per dollar is 5 cents. Find the largest net income possible.

Solution by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

Let x = the number of dollars, gross income; y = the rate of taxation on \$1; xy = the tax on x dollars; and u = the net income; from which we have $u=x-xy$... (1).

From the problem we have, $(8000)^{\frac{1}{3}} : x^{\frac{1}{3}} :: 0.05 : y$; or $y=x^{\frac{1}{3}}/400$... (2); and from (2), $xy=x^{\frac{4}{3}}/400$... (3). Substitute in (1) the value of xy from (3), and we have, $u=x-x^{\frac{4}{3}}/400$... (4). Differentiate (4) and we have, $du/dx=1-x^{\frac{1}{3}}/300=0$; or $x=\$27{,}000{,}000$. Substitute this value of x in (2), and we have, $y=\frac{3}{4}$, the rate of taxation. Therefore the net income is $\frac{1}{4}$; or $\frac{1}{4}$ of \$27,000,000 = \$6,750,000, the maximum net income.

To find when the rate of taxation is prohibitive, let z = the gross income, and we have, $(8000)^{\frac{1}{3}} : z^{\frac{1}{3}} :: 5\% : 100\%$; or $z=\$64{,}000{,}000$. That is to say, when the gross income is \$64,000,000, the whole income is taken for the tax.

Also solved by H. C. Feemster, J. Scheffer, J. E. Sanders, V. M. Spunar, S. A. Corey, J. W. Clawson, and the Proposer.

MECHANICS.

226. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

A frustum of a cone, vertical angle a, is cut off by two spheres whose centers are the vertex. The radius of one sphere is n times that of the other, and the density of the cone varies as the distance of the vertex. Find the ratio into which the centroid of the frustum divides the axis.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let R=radius of smaller sphere, nR=radius of larger sphere. Take the axis of the cone as that of z. Then $\bar{x}=\bar{y}=0$.

$$\bar{z}=\frac{\int\int\int r^4\sin\theta\cos\theta\,d\theta\,d\phi\,dr}{\int\int\int r^3\sin\theta\,d\theta\,d\phi\,dr}.$$

By frustum of cone here is evidently meant that part of the cone between the spheres.

$$\therefore \bar{z}=\frac{\int_0^{\frac{1}{2}a}\int_0^{2\pi}\int_R^{nR} r^4\sin\theta\cos\theta\,d\theta\,d\phi\,dr}{\int_0^{\frac{1}{2}a}\int_0^{2\pi}\int_R^{Rn} r^3\sin\theta\,d\theta\,d\phi\,dr}=\tfrac{4}{5}R\left(\frac{n^5-1}{n^4-1}\right)\frac{\int_0^{\frac{1}{2}a}\sin\theta\cos\theta\,d\theta}{\int_0^{\frac{1}{2}a}\sin\theta\,d\theta}$$

$$=\tfrac{2}{5}R\left(\frac{n^5-1}{n^4-1}\right)(1+\cos\tfrac{1}{2}a).$$

$$\frac{nR-\bar{z}}{\bar{z}-R}=\frac{5n(n^4-1)-2(n^5-1)(1+\cos\frac{1}{2}a)}{2(n^5-1)(1+\cos\frac{1}{2}a)-5(n^4-1)}.$$

If planes were passed through the intersections of the spheres with the cone, then $R\cos\frac{1}{2}a\,(n-1)$ is the height of the frustum, and the limits of r are $nR\cos\frac{1}{2}a\sec\theta=R_1$, and $R\cos\frac{1}{2}a\sec\theta=R_2$.

$$\therefore \bar{z}=\tfrac{4}{5}R\cos\tfrac{1}{2}a\left(\frac{n^5-1}{n^4-1}\right)\frac{\int_0^{\frac{1}{2}a}\sin\theta\cos\theta\sec^5\theta\,d\theta}{\int_0^{\frac{1}{2}a}\sin\theta\sec^4\theta\,d\theta}=\tfrac{4}{5}\left(\frac{n^5-1}{n^4-1}\right)R\cos\tfrac{1}{2}a.$$

$$\therefore \frac{nR\cos\frac{1}{2}a-\bar{z}}{\bar{z}-R\cos\frac{1}{2}a}=\frac{n^5-5n+4}{4n^5-5n^4+1}.$$

II. Solution by J. A. CAPARO, C. E., Notre Dame University, Notre Dame, Ind.

Let α=one-half the vertical angle of cone, a, and an radii of spheres. With the vertex of the cone as origin, and the axis of y as the axis of the cone, the mass of the frustum contained between the spheres is

$$M=k\pi\tan^2\alpha\int_{a\cos\alpha}^{na\cos\alpha}y^3dy+k\pi\int_{an\cos\alpha}^{an}(a^2-y^2)ydy-k\pi\int_{a\cos\alpha}^{a}(a^2-y^2)ydy$$

$$=\frac{\pi ka^4(n^4-1)\sin^2\alpha}{4}.$$

These integrals represent the mass of a frustum whose bases are planes passing through the line of intersection of the surfaces, and the masses of the spherical segments added and subtracted from the above mass. Call these masses M_c, M_a, M_n, and $\bar{y}_c$, $\bar{y}_a$, $\bar{y}_n$, distances of their centroids from origin.

Then, $M_c\bar{y}_c=\pi k\tan^2\alpha\int_{a\cos\alpha}^{na\cos\alpha}y^4dy=\frac{1}{5}\pi ka^5\cos^5\alpha\tan^2\alpha(n^6-1)$.

$$M_a\bar{y}_a=\pi k\int_{a\cos\alpha}^{a}(a^2-y^2)y^2dy=\frac{\pi ka^5}{15}(2-5\cos^3\alpha+3\cos^5\alpha).$$

$$M_n\bar{y}_n=\pi k\int_{an\cos\alpha}^{an}(a^2-y^2)y^2dy=\frac{\pi ka^5n^3}{15}[5(1-\cos^3\alpha)-3n^2(1-\cos^5\alpha)].$$

But $M\bar{y}=M_c\bar{y}_c+M_n\bar{y}_n-M_a\bar{y}_a$.

$$\therefore\bar{y}=\frac{4a[2+5n^3-3n^5+n^3\cos^3\alpha(3n^2+5)+2\cos^3(4+3\cos^2\alpha)]}{15(n^4-1)\sin^2\alpha}.$$

227. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Regarding the earth as a homogeneous sphere, radius R, acceleration at the surface g, investigate the motion of a sphere, radius b, moving through a straight tunnel between two points on the surface not diametrically opposite.

Solution by the PROPOSER.

Let $AB=2a$ be the tunnel; $OA=OB=R$; C the mid-point of AB; P and R, points between A and C; $OP=y$, $PC=x$, $CR=d$, $\angle POC=\theta$; f the acceleration at P; f' the acceleration along PC at P, and R the starting point of the sphere. Beneath the earth's surface, the acceleration varies directly as the distance from the center.

$\therefore f:g=y:R$ or $f=gy/R$. $f':f=\sin\theta:1$ or $f'=f\sin\theta=gy\sin\theta/R$.

But $y=x/\sin\theta$. $\therefore f'=gx/R$.

I. If the tunnel is perfectly smooth, the equation of motion is $d^2/dt^2 + gx/R = 0$. $\therefore (dx/dt)^2 + gx^2/R = C$. Since $t=0$ when $x=d$, $C = gd^2/R$.

$\therefore (dx/dt)^2 = (g/R)(d^2 - x^2) = (\text{velocity})^2 = v^2$. When the sphere arrives at C, $x=0$. $\therefore v = d\sqrt{(g/R)} = a\sqrt{(g/R)}$. When the sphere starts at A,

$$t = \sqrt{\frac{R}{g}} \int_0^m \frac{dx}{\sqrt{(d^2 - x^2)}} = \sqrt{\frac{R}{g}} \sin^{-1} \frac{m}{d}. \quad \text{If } m=d,\ t = \frac{\pi}{2}\sqrt{\frac{R}{g}}.$$

As this expression is independent of the distance from C, the time for the sphere to move from A to C or from any point between A and C to C is the same. The time is the same for any tunnel, and is the same as the time required for the sphere to fall from the surface to the center, or from any point beneath the surface to the center of the earth.

Let $R = 20902410$ feet, $g = 32.10614$ feet.

$\therefore \sqrt{(R/g)} = 806.871$, and therefore $t = (\frac{1}{2}\pi)\sqrt{(R/g)} = 1267.433$ seconds $= 21$ minutes, 7.433 seconds.

II. When the tunnel is perfectly rough. Let F be the friction. Then the equations of motion are $d^2x/dt^2 + gx/R + F = 0$, $k^2 d^2\phi/dt^2 = bF$, $b\phi = x$ or $bd\,\phi = dx$, $bd^2\phi = d^2x$.

$$\therefore \left(\frac{k^2}{b^2}\right)\frac{d^2x}{dt^2} = F; \quad \therefore \left(\frac{b^2+k^2}{b^2}\right)\frac{d^2x}{dt^2} + \frac{gx}{R} = 0; \quad \frac{d^2x}{dt^2} + \frac{5gx}{7R} = 0; \quad \left(\frac{dx}{dt}\right)^2 = v^2 = \frac{5g}{7R}(d^2 - x^2).$$

$\therefore v = d\sqrt{(5g/7R)}$, at the point C; $v = a\sqrt{(5g/7R)}$, when the sphere starts at A.

$$t = \sqrt{\frac{7R}{5g}} \int_0^d \frac{dx}{\sqrt{(d^2 - x^2)}} = \frac{\pi}{2}\sqrt{\frac{7R}{5g}} = 24 \text{ minutes, } 59.627 \text{ seconds.}$$

Therefore the time is the same for any perfectly rough tunnel.

If the sphere starts with a velocity v_1^2, then for a perfectly smooth tunnel,

$$v^2 = \left(\frac{dx}{dt}\right)^2 = \frac{g}{R}(a^2 - x^2) + v_1^2,$$

supposing the sphere to start from A with the velocity v_1.

$$\therefore t = \sqrt{R} \int_0^a \frac{dx}{\sqrt{[Rv_1^2 + g(a^2 - x^2)]}} = \sqrt{\frac{R}{g}} \sin^{-1} \frac{a\sqrt{g}}{\sqrt{(Rv_1^2 + ga^2)}}.$$

This equals the time to go from A to C.

For a perfectly rough tunnel, $v^2=(dx/dt)^2=(5g/7R)(a^2-x^2)+v_1^2$.

$$\therefore t=\sqrt{7R}\int_0^a \frac{dx}{\sqrt{[7Rv_1^2+5g(a^2-x^2)]}}=\sqrt{\frac{7R}{5g}}\sin^{-1}\frac{a\sqrt{5g}}{\sqrt{(7Rv_1^2+5ga^2)}}.$$

III. If the tunnel extends beyond the surface of the earth: then let SB be the tunnel. Above the earth's surface, acceleration varies inversely as the square of the distance from the center.

Let $CO=c$, $OQ=y$, $CQ=x$, $SC=h$. Then $f:g=R^2:y^2$ or $f=gR^2/y^2=gR^2/(c^2+x^2)$. $f':f=\sin\theta:1$. $\therefore f'=f\sin\theta=fx/\sqrt{(c^2+x^2)}$.

$$\therefore f'=\frac{gR^2x}{\sqrt{[(c^2+x^2)^3]}}. \quad \therefore \frac{d^2x}{dt^2}+\frac{gR^2x}{\sqrt{[(c^2+x^2)^3]}}=0.$$

$$\therefore \left(\frac{dx}{dt}\right)^2=2gR^2\left[\frac{1}{\sqrt{(c^2+x^2)}}-\frac{1}{\sqrt{(c^2+h^2)}}\right]=v_1^2.$$

$$\therefore v_1=R\sqrt{2g\left[\frac{1}{\sqrt{(a^2+c^2)}}-\frac{1}{\sqrt{(h^2+c^2)}}\right]}=R\sqrt{\frac{2g}{R}-\frac{2g}{\sqrt{(c^2+h^2)}}}.$$

When the sphere reaches A,

$$t=\frac{\sqrt[4]{(c^2+h^2)}}{R\sqrt{(2g)}}\int_0^h \frac{\sqrt[4]{(c^2+x^2)}\,dx}{\sqrt{[\sqrt{(c^2+h^2)}-\sqrt{(c^2+x^2)}]}}.$$

$$\therefore t=\frac{2(c^2+h^2)^{\frac{1}{2}}}{R\sqrt{(2g)}}\int_{\phi_1}^{\frac{1}{2}\pi}\frac{\sin^4\psi\,d\psi}{\sqrt{[(c^2+h^2)\sin^4\psi-c^2]}},$$

where $c^2+x^2=(c^2+h^2)\sin^4\psi$ and $\phi_1=\sin^{-1}\sqrt{\frac{R^2}{c^2+h^2}}$.

t=time to move from S to A. For a perfectly rough tunnel, $v_2=\sqrt{(\frac{5}{7})}v_1$; $t_1=\sqrt{(\frac{7}{5})}t$.

Also solved by S. Lefseletz.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

160. Proposed by H. S. VANDIVER, Bala, Pa.

Prove that the integer next above $(1+\sqrt{3})^{2n}$ is divisible by 2^{n+1}.

Proof by G. E. WAHLIN, Urbana, Ill.

$|1-\sqrt{3}|<1$. Hence (1) $0<(1-\sqrt{3})^{2n}<1$.

But $(1+\sqrt{3})^{2n}+(1-\sqrt{3})^{2n}=E$, a rational integer, and since (1) is true, this is the integer next above $(1+\sqrt{3})^{2n}$.

Since $\frac{(1\pm\sqrt{3})^2}{2}=2\pm\sqrt{3}$, we have $\frac{(1\pm\sqrt{3})^{2n}}{2^n}=(2\pm\sqrt{3})^n=P\pm Q\sqrt{3}$, P and Q rational integers.

Then $\frac{(1+\sqrt{3})^{2n}}{2^n}+\frac{(1-\sqrt{3})^{2n}}{2^n}=\frac{E}{2^n}=2P$. Therefore, $\frac{E}{2^{n+1}}=P$, and hence E is divisible by 2^{n+1}.

Also solved by J. Scheffer, G. B. M. Zerr, and V. M. Spunar.

PROBLEMS FOR SOLUTION.

ALGEBRA.

321. Proposed by C. C. BLAND, Attorney at Law, Rolla, Mo.

A corporation is capitalized for $20,000. 125 shares of the par value of $100 per share has been issued. A has 27 19/78 shares. B, C, D, E and F each have 19 43/78 shares. It is the wish of the corporation to cancel the certificates held by A, B, C, D, E, and F, and to issue new certificates to each of them in lieu of those now held by them, and to avoid the issuance of any certificate for a fraction of a share. How many shares should each receive, the whole not to exceed 200, at the same time maintaining the present interest of each in the corporation?

322. Proposed by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

Take six consecutive prime numbers, as 53, 59, 61, 67, 71, and 73, and find the least whole number such that if it be divided by 59 the remainder will be 53, if it be divided by 67 the remainder will be 61, and if it be divided by 73 the remainder will be 71, and show that this least whole number and the succeeding consecutive whole numbers that will fulfill this condition as to divisions and remainders are in arithmetical progression; and also show whether or not this is a general law for n consecutive prime numbers; and if there be such a general law whether or not that general law will lead to a general law for the finding of prime numbers.

323. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

Show that the relation $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$ can hold for real numbers only when they are in proportion.

GEOMETRY.

348. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

Two parabolas and a rectangular hyperbola circumscribe a given quadrilateral. Find a relation between the squares of the latera recta of the parabolas and the squares of the perpendiculars from the center of the hyperbola to the axes of the parabolas.

349. Proposed by J. A. CAPARO, Notre Dame University, Notre Dame, Indiana.

Given the radius of a circular smooth cylinder and its position with respect to a source of light and the eye. Find a geometrical construction to determine the line of brilliancy.

350. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Given the quadrilateral $AB=a=225$, $BC=b=153$, $CD=c=207$, $DA=d=135$, $AC=e=240$. Find the side of the square inscribed in this quadrilateral having a corner in each side.

CALCULUS.

279. Proposed by L. H. McDONALD, M. A., Ph. D., Sometime Tutor at Cambridge, Jersey City, N. J.

Find the ellipse of minimum area which will pass through the vertices of a triangle. (Hedrick-Goursat's *Math. Anal.*, p. 133, ex. 9.)

280. Proposed by C. N. SCHMALL, 89 Columbia Street, New York.

Find the envelope of the system of spheres

$$\left.\begin{aligned}(x-a)^2+(y-b)^2+z^2&=r^2\\ a^2+b^2&=c^2\end{aligned}\right\}.$$

281. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that if n be a positive integer greater than unity,

$$\text{Log}n=(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots+\frac{1}{n-1}+\frac{1}{2n})-C+\frac{B_1}{2n^2}-\frac{B_2}{4n^4}+\frac{B_3}{6n^6}-\frac{B_4}{8n^8}+\ldots(1).$$

REMARK.—This development may be obtained by employing the formula given by the proposer in *Annals of Mathematics*, second series, Vol. 5, No. 4, July, 1904, but other proofs are also desired. It will be observed that (1) offers a ready method of evaluating C, which is remarkably simple and very rapidly convergent if $n>$ or $=10$. Compare with method given by Mr. Bromwitch in *Messenger of Mathematics*, October, 1906.

MECHANICS.

233. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

A shot, mass m, is fired from a gun, mass M, which can move on a horizontal plane. The muzzle velocity of the shot is given, the muzzle always points in the same direction as the shot leaves it. The envelope of the trajectories is a parabola. Find the ratio of the distances of its focus and directrix from the plane in terms of M and m.

234. Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.

A ladder is placed with one end resting against a smooth wall and making with it an angle ϕ. Also, the roughness of the ground prevents it from slipping. A man weighing as much as the ladder ascends to the top. Taking μ as the coefficient of friction, prove:

(a) The ladder will slip before he gets to the top if $\phi>\tan^{-1}4\mu/3$.

(b) If the ascent be feasible, there will be three times as much friction when he is at the top as when he is at the bottom, (See Jeans' *Theoretical Mechanics*, p. 47.)

235. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

A uniform heavy rod turns freely round a hinge at one end and rests with the other against a rough vertical wall, at angle, α, to the wall. Find the angle of arc on which this end may rest, and the pressures at the ends of the arc.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

165. Proposed by J. EDWARD SANDERS, Weather Bureau, Chicago, Ill.

Factor (if possible), 11, 111, 111, 111.

166. Proposed by H. S. VANDIVER, Bala, Pa.

Eliminate any five of the seven quantities h, j, k, l, m, r, s, from:

$$j+m+r=k+l+s,$$
$$h=3(r+s)-(n+1),$$
$$j+m+2r+s=n,$$
$$hr+j^2+km+ms=jk+mr+s^2+rs,$$
$$ks+jl+rs+r^2=sh+jk+l^2+rk,$$
$$hr+jl+ks+ls+km=jm+mr+2rs+rj.$$

167. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Prove that $\Pi \frac{p+(-1)^{\frac{1}{2}(p+1)}}{p-(-1)^{\frac{1}{2}(p+1)}}=2$, the consecutive values of p being the natural odd primes in order.

AVERAGE AND PROBABILITY.

204. Proposed by F. P. MATZ, Reading, Pa.

On a random chord in a circle two points are taken at random. What is the chance a second chord drawn at random will pass between the two points?

205. Proposed by J. EDWARD SANDERS, Weather Bureau, Chicago, Ill.

What is the probability that the triangle formed by joining three points, one at random in each of three equal circles, mutually tangent, has an obtuse angle?

NOTES AND NEWS.

Professor L. E. Dickson left on September 4th for Europe, where he will spend the year in study and travel. F.

Professor R. D. Carmichael has been granted a year's leave of absence and is now studying mathematics at Princeton. F.

Professor B. F. Yanney, of Mount Union College, Alliance, Ohio, has been granted a year's leave of absence. He will spend the year in study at the University of Chicago. F.

Professor W. A. Manning of Stanford University, secretary of the San Francisco Section of the American Mathematical Society, has joined the mathematical faculty of the University of Illinois for one year in exchange with E. W. Ponzer. Professor Manning is the author of various articles on substitution groups. M.

On July 11th occurred the death of Professor Simon Newcomb, probably the most distinguished astronomer that America ever produced. Professor Newcomb was also recognized as a mathematician of the first rank and was honored the world over for his great scientific achievements. For a biography of Professor Newcomb, see THE AMERICAN MATHEMATICAL MONTHLY, Vol. I, pp. 253-256. F.

The seventy-ninth annual meeting of the British Association for the Advancement of Science was held at Winnipeg, Canada, under the presidency of J. J. Thomson from August 25 to September 1, 1909. Fourteen hundred members and associates were in attendance, about five hundred coming from Europe and about one hundred and fifty from the United States. The papers devoted to pure mathematics were as follows:

E. H. Moore, Theorems in General Analysis; E. W. Hobson, F. R. S., On the Present Position of the Theory of Aggregates; G. A. Miller, Generalizations of the Icosahedral Group; G. A. Bliss, A New Proof of Weierstrass' Theorem; J. H. Grace, F. R. S., On Ideal Numbers; P. A. MacMahon, F. R. S., On a Correspondence in the Theory of the Partition of Numbers; W. H. Metzler, On a Continuant Expressed as the Product of Linear Factors; Ellery W. Davis, Imaginary Geometry of the Conic; Florian Cajori, On the Invention of the Slide Rule; J. W. Nicholson, The Asymptotic Expansion of Legendre Functions. M.

BOOKS.

Coordinate Geometry. By Henry Burchard Fine and Henry Dallas Thompson. 8vo. Red Cloth Sides and Leather Back. viii+300 pages. New York: The Macmillan Co.

This book, which has the advantage of having been tested out in class room use for three years, will appeal strongly to all good teachers of Analytical Geometry because of the scholarly presentation of the subject. While it does not begin with the easiest possible concepts leading up to the subject, yet it is believed to be sufficiently elementary in its initial steps to enable the earnest student to easily master it. The treatment of higher plane curves is very brief, and it is to be regretted that brief historical notes concerning those that are treated have been omitted.

Excellent engravings of some of the second degree surfaces are given in the last pages of the book. F.

An Elementary Treatment of the Theory of Spinning Tops and Gyroscopic Motion. By Harold Crabtree, M. A., Formerly Scholar of Pembroke College, Cambridge. Assistant Master at Charterhouse. With Illustrations. 8vo. Cloth, xii+140 pages. Price, $1.50. New York: Longmans, Green & Co.

The author's object in writing this interesting book is to bring within the range of the abler mathematicians of the public schools of England and of the First Year Undergraduates of the universities, a subject which has been previously considered too difficult for any but the more advanced students in mathematics.

The story was told that when Felix Klein gave his lecture on the Top at Princeton, a

lady told Professor Klein that her little boy (about ten years old) was greatly interested in tops, and she thought he would enjoy a lecture on the subject from so distinguished a scholar. The reporter failed to tell how the youngster entertained himself during the lecture, but if he did not have to be aroused from his slumbers by some vigorous shaking at its close he surely must have been a very unusual lad.

The book before us is not intended, it must be understood, for popular reading, nor for the amusement of boys interested in tops. It does, however, put the theory of tops in a mathematical dress easily recognized by any one having a good working knowledge of the Calculus, and a clear understanding of a few fundamental laws of Mechanics. The book is well written, and will be of interest to the inventive mind as well as to the mathematician. F.

A Treatise on Differential Geometry. By Luther Pfahler Eisenhart, Preceptor in Mathematics, Princeton University. 8vo. Cloth. 474 pages. With Diagrams. Price, $4.50. Boston and Chicago: Ginn & Co.

It is the purpose of this book to introduce the student to the methods of differential geometry and to the theory of curves and surfaces developed thereby, to such an extent that he will be prepared to read the most extensive foreign treatises and journal articles. The reader is suppose to possess a knowledge of the calculus, elementary differential equations, and the elements of coordinate geometry of three dimensions. Hence the first half of the book may be used with seniors, and the remainder will constitute a full-year course for graduate students.

The method generally used is that of Gauss, common among German and Italian writers, but the kinematical method, frequently adopted in France, has been developed and applied where more feasible. This has been done not only because it furnishes the student with a powerful operator, but for the reason, also, that it develops geometrical thinking.

There are several hundred problems, some of which are direct applications of the accompanying sections, but many are theorems which might properly be established in a more extensive treatise. These have been inserted as an incentive to research and as preparation for larger problems.

The Integrals of Mechanics. By Oliver Clarence Lester, Professor of Physics in the University of Colorado; formerly Instructor in Physics in the Sheffield Scientific School, Yale University. 8vo. Cloth, 67 pages. With Diagrams. Price, 80 cents. Boston and Chicago: Ginn & Co.

The aim of this book is to furnish the conclusion to courses in the Integral Calculus such as are usually given in colleges and technical schools, and at the same time to provide for the beginning of Theoretical Mechanics, which usually follows the Calculus. The subject-matter is concerned entirely with such applications of the Calculus as the calculation of lengths, areas, volumes, densities, centers of mass, moments of inertia, and ellipsoids of inertia. These subjects are treated in great detail, all principles being fully illustrated by examples worked out in the text and by numerous problems set as exercises.

Since the ground covered is common to both the Integral Calculus and to Theoretical Mechanics, the author hopes in this way to save both time and energy; to save time by providing applications of the Calculus useful in Mechanics; to save energy by treating the purely mathematical parts of Mechanics entirely apart form the ideas of force and motion. This method avoids breaks in the continuity of the Mechanics course proper, and minimizes the liability of the student to such troublesome confusions as moment of inertia with the moment of a force, or center of gravity with the force of gravity.

Vector Analysis. An Introduction to Vector-Methods and Their Various Applications to Physics and Mathematics. By Joseph George Coffin, B. S., Ph. D., Instructor in Physics at the College of the City of New York. 12mo, Cloth. xix+248 pages, 69 figures. Price, $2.50 net. New York: John Wiley & Sons.

The first part of the book is devoted to a concise treatment of the fundamental principles of the subject, the remaining chapters, to the application of the analysis to the *beginnings* of mathematical physics, including geometry, mechanics, magnetism, electricity, heat and hydrodynamics. It was found necessary to omit many beautiful applications in elasticity, electron theory and other parts of physics in order to keep the size of the volume within bounds.

The student who takes up the later chapters, is supposed to be familiar, to a certain extent, with the subjects therein contained, and these chapters are intended to show the beginner how to translate and demonstrate the theorems into the new calculus. The writer therefore makes this his apology for a certain necessary lack of logical sequence in the treatment of the various subjects.

The treatment of alternating currents and allied subjects has been omitted, because in practically every modern book on the subject of notation of the special vector method employed, is fully explained in some part of it.

The notation adopted is that of Prof. Willard Gibbs, one of the too few great American physicists and mathematicians. The reasons leading to this choice are fully set forth in the Appendix. From *Preface.*

Complete Arithmetic. By George Wentworth and David Eugene Smith. 12mo. Cloth. Illustrated. vi+474 pages. Price, 60 cents. New York and Chicago: Ginn & Co.

The *Complete Arithmetic* is a text-book for grammar-school grades, thoroughly modern in spirit and material and arranged according to a topical plan.

The keynote of the method—present the reason briefly but clearly, then furnish such an amount of practice that the pupil cannot forget the principle—is that which has made the Wentworth texts the standard for a generation. Theory is reduced to a minimum and practice is abundantly provided in more than six thousand carefully graded problems and examples, all absolutely new.

The work conforms with modern business customes, and the needs of the future citizen are constantly kept in mind. The topics properly close with sets of exercises that relate to the vocational interests of our country, to the end that pupils may leave the *study* of arithmetic with the real *applications* of arithmetic clearly in mind.

The book contains all the topics ordinarily studied after a primary arithmetic has been completed, and omits such subjects as are too technical or have become obsolete.

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THE FOURTH DIMENSION.

By E. J. WILCZYNSKI, University of Illinois.

In order to describe the position of a point upon a line, we usually proceed as follows. We place a scale, like that of a thermometer, next to the line and make note of the scale-reading which corresponds to the point considered. As in the case of the thermometer this scale reading will be positive or negative according as the point is situated on one side or the other of the zero point of the scale. It thus becomes possible to characterize the position of a point of a line by means of a *single* positive or negative number. This is what is meant by the statement that a line has only *one dimension.*

If we wish to describe the position of a point on a surface, we shall have to employ two numbers. A familiar illustration of this fact is given by the ordinary method of describing the position of a point on the earth's surface by means of its latitude and longitude. These two numbers are called the coordinates of the point. The fact that *two* numbers are required is equivalent to the statement that the surface has *two dimensions.*

But if we take into account the fact that there are mountains and valleys, it becomes clear that a third coordinate becomes necessary which shall express the altitude of a point above, or its depression below sea-level. Obviously three numbers will suffice to determine the position of any point in space. This fact, expressed in another way, leads to the statement that *space has three dimensions.*

Nevertheless, this property of space of having three dimensions is not really a property of space at all, but depends upon our way of looking at it which, as we shall see, is largely due to physiological causes and has no logical foundation whatever. In fact, a number applied to an object of experience always implies that something has been taken as a unit. If we say that the length of a line is expressed by the number five, we can attach no meaning to such an assertion unless we know what unit of measurement has been employed. In the same way, the statement that space has three dimensions is incomplete and meaningless, although our above discussion may